

## The algebraic approach to the Morse oscillator

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1980 J. Phys. A: Math. Gen. 13 773

(<http://iopscience.iop.org/0305-4470/13/3/010>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.252.86.83

The article was downloaded on 31/05/2010 at 04:46

Please note that [terms and conditions apply](#).

## The algebraic approach to the Morse oscillator

M Berrondo†§|| and A Palma‡¶

† Institute for Advanced Studies, The Hebrew University, Jerusalem, Israel

‡ Instituto Mexicano del Petróleo, Av Cien Metros 500, México 14, DF, México

Received 24 April 1979, in final form 30 June 1979

**Abstract.** The discrete spectrum for a Morse oscillator is found using an  $SO(2, 1)$  algebra. Since this algebra does not prove to be appropriate to compute matrix elements for the oscillator eigenfunctions, we construct a  $B_1$ -type algebra with the aid of an auxiliary angle variable. Matrix elements and recurrence relations are found for several useful operators using this algebra. The limit in which the anharmonicity tends to zero is studied.

### 1. Introduction

In this paper we present the solution for the bound-state problem of a Morse oscillator in one dimension (Morse 1929) using two different Lie algebras. Many quantum-mechanical problems have been studied using operator methods (Wybourne 1974), in particular, the harmonic oscillator (Louisell 1973), the Coulomb problem (Englefield 1972, Barut and Kleinert 1967, Bednář 1973) and angular momentum (Edmonds 1957). Little attention, however, has been paid to the Morse potential well, in spite of its practical importance in analysing anharmonic vibrations (Pauling and Wilson 1935, Huffacker 1976), molecular energy transfer (Rapp and Kassal 1969) and atom–surface scattering (Cabrera *et al* 1970). The algebraic approach is proving to be a basic tool in the study of inelastic collisions (Alhassid and Levine 1978). Hence a thorough understanding of the time honoured Morse oscillator in algebraic terms seems appropriate.

Ladder operators for the bound states have been known since the exhaustive work of Infeld and Hull (1951), and some of their properties have been exploited in vibrational problems (Huffacker and Dwivedi 1975). These ladder operators were first obtained with the aid of the factorisation method (Infeld and Hull 1951). They depend explicitly on the vibrational quantum number, so they are well defined as long as they act upon a specified eigenstate.

As a generalisation of the two different types of factorisation (types B and F) (Infeld and Hull 1951), we obtain in this work two different Lie algebras for the Morse oscillator. The first one is a realisation of an  $SO(2, 1)$  algebra. It follows from a B-type factorisation. Indeed, the Schrödinger equation for the Morse potential is equivalent (Morse 1929) to the radial equation of the Coulomb problem. This in turn is associated with an  $SO(2, 1)$  algebra (Armstrong 1970, 1971, Čížek and Paldus 1977). It is found,

§ On leave from Instituto de Física, University of México, Apdo Postal 20–364, México 20, DF, México.

|| Consultant at Instituto Mexicano del Petróleo.

¶ Also at Escuela Superior de Física y Matemáticas (IPN).

however, that the ladder operators in this algebra shift the well's depth at *constant energy* (Huffacker and Dwivedi 1975). In other words, two eigenstates of the Morse oscillator correspond to different irreducible representations (IR) of the group  $SO(2, 1)$ . Therefore, matrix elements between different energy states are awkward to compute in this scheme.

A different approach to the problem consists of introducing an auxiliary variable (Armstrong 1970, 1971, Crubellier and Feneuille 1974, Miller 1968) in order to build an  $SO(3)$  algebra or, more properly, a  $B_1$ -type algebra (Miller 1968). This approach corresponds to the F-type factorisation (Infeld and Hull 1951). The associated ladder operators shift the vibrational quantum number by one unit. We are then able to compute matrix elements for different bound states. The auxiliary variable can be taken as a dummy angle ranging from 0 to  $2\pi$ . Alternatively, it may be interpreted as a dilated time (Anderson *et al* 1973) when working in the Schrödinger picture (Chacón *et al* 1976). The time dilation is introduced in order to 'linearise the energy spectrum' (Anderson *et al* 1973). The resulting operators include (partial) derivatives with respect to the dilated time. A similar  $B_1$  algebra can be defined for the hydrogen atom problem. There the ladder operators will have the effect of shifting the angular momentum in the radial equation (Herrick and Sinanoğlu 1972).

Finally, if we let the anharmonicity parameter in the Morse potential go to zero, we recover the harmonic oscillator. In this case, the algebra is contracted into the algebra of the harmonic oscillator as we show below.

In § 2, we define the  $SO(2, 1)$  algebra for the Morse potential, following the constructive approach of Čížek and Paldus (1977). Using this algebra, we derive the discrete spectrum, with the aid of its Casimir operator. In § 3, we make use of the auxiliary variable to obtain the  $B_1$  algebra. We then show how the matrix elements for different operators can be calculated. We also discuss the hermiticity properties of the ladder operators in the physical metric. Section 4 shows how the harmonic limit is attained. Finally, in § 5 we present a brief comparison between the two algebras, and point out some further developments.

## 2. The discrete spectrum

The Morse potential is defined in terms of two constants, the potential depth  $D$  and an inverse distance  $a$ :

$$V(u) = D e^{-2au} - 2D e^{-au}. \quad (2.1)$$

Here we have chosen the origin  $u = 0$  as the equilibrium distance. The potential tends to zero asymptotically as  $U \rightarrow \infty$ , so its minimum is  $V(u = 0) = -D$ . Throughout this work we shall take equation (2.1) as a one-dimensional potential (Morse 1929, Pauling and Wilson 1935). This implies that the eigenfunctions vanish at  $u \rightarrow \pm\infty$ .

It is convenient to define a dimensionless distance  $x = au$  and to introduce the (inverse) anharmonicity constant  $K$ :

$$K = \sqrt{2m_0 D} / \hbar a, \quad (2.2)$$

given in terms of the reduced mass  $m_0$ . The Schrödinger equation for  $S(x)$  is then

$$(d^2 S(x)/dx^2) + [\epsilon + K^2(2e^{-x} - e^{-2x})]S(x) = 0 \quad (2.3)$$

where  $\epsilon$  is the dimensionless energy  $\epsilon = 2m_0E/\hbar^2 a^2$ . Choosing  $S(x)$  as a normalised function, we have that

$$\int_{-\infty}^{\infty} S_n^*(x)S_m(x) dx = \delta_{nm} \tag{2.4}$$

for any bound states  $n$  and  $m$ .

Equation (2.3) is transformed into a simpler form with a further change of variable and function (Morse 1929):

$$\begin{aligned} y &= K e^{-x} & (\infty > y \geq 0) \\ \psi(y) &= e^{-x/2} S(x). \end{aligned} \tag{2.5}$$

The equation for  $\psi(y)$  is

$$d^2\psi(y)/dy^2 + [(\epsilon + \frac{1}{4})y^{-2} + 2Ky^{-1} - 1]\psi(y) = 0, \tag{2.6}$$

which can be rewritten as an eigenvalue equation for  $K$  (cf Bednář 1973 for the Coulomb case). Introducing the canonical momentum  $p = -id/dy$ , we get

$$\frac{1}{2}[yp^2 - (\epsilon + \frac{1}{4})y^{-1} + y]\psi = K\psi. \tag{2.7}$$

In order to define the proper  $SO(2, 1)$  algebra, we first notice that the operators  $y$ ,  $yp$  and  $yp^2$  close under commutation (Čížek and Paldus 1977). Since adding a term proportional to  $y^{-1}$  to the latter does not change the commutators, we write

$$W_1 = y \quad W_2 = yp \quad W_3 = yp^2 - [(\epsilon + \frac{1}{4})/y]. \tag{2.8}$$

The relations

$$[W_1, W_2] = i W_1 \quad [W_2, W_3] = i W_3 \quad [W_1, W_3] = 2 i W_2 \tag{2.9}$$

can be obtained directly from the canonical one  $[y, p] = i$ .

Comparing (2.8) with the eigenvalue equation (2.7), we define

$$T_3 = \frac{1}{2}(W_1 + W_3) \tag{2.10a}$$

and the corresponding ladder operators are

$$T_{\pm} = \frac{1}{2}(W_3 - W_1) \pm i W_2. \tag{2.10b}$$

From (2.8) it indeed follows that  $\{T_3, T_+, T_-\}$  form an  $SO(2, 1)$  algebra:

$$[T_3, T_{\pm}] = \pm T_{\pm} \quad [T_+, T_-] = -2T_3. \tag{2.11}$$

We see immediately that the ladder operators  $T_{\pm}$  shift the value of  $K \rightarrow K \pm 1$  for a given energy  $\epsilon$ . They change the potential parameters, but not the energy. The corresponding Casimir operator is

$$C^2 = T_3^2 \mp T_3 - T_{\pm}T_{\mp} = -\epsilon - \frac{1}{4}. \tag{2.12}$$

Using (2.12) and the commutation relations (2.11), we can determine the energy spectrum: for any given (negative) energy  $\epsilon$ , there is a minimum value of  $K$  for which such an eigenfunction exists. Calling this value  $K_0$ , and the corresponding eigenfunction  $\psi_0$ , we have that

$$T_3\psi_0 = K_0\psi_0 \quad T_-\psi_0 = 0. \tag{2.13}$$

Since  $T_+\psi_K \propto \psi_{K+1}$ , the eigenvalues of  $T_3$  are given by  $K = K_0 + n$ , where  $n = 0, 1, 2, \dots$ . From (2.12) and (2.13), we then get

$$(T_3^2 - T_3 - C^2)\psi_0 = [(K_0 - \frac{1}{2})^2 + \epsilon]\psi_0 = 0, \quad (2.14)$$

which finally yields

$$\epsilon_n = -(K - n - \frac{1}{2})^2 \quad (2.15)$$

in terms of the actual  $K$ . For fixed  $K$ ,  $n = 0, 1, 2, \dots, (K - \frac{1}{2})$ , so that the largest integer in  $K + \frac{1}{2}$  gives the number of bound states. If  $K$  is smaller than  $\frac{1}{2}$ , the potential does not support any bound state (Morse 1929).

Although the problem is similar to the hydrogen atom, there is an important difference. In the Coulomb problem, in order to rewrite the equation as an eigenvalue equation for the charge, it is necessary to make a space dilation  $r \rightarrow r/n$  which depends on the energy through the quantum number  $n$ . This implies different dilations for different eigenstates. For the Morse oscillator, however, the necessary dilation depends only on the potential parameters. It is implicitly introduced in equation (2.5):  $e^{-x} \rightarrow K e^{-x}$ . In fact, in terms of the variable  $x$ , it reduces to a (constant) translation  $x \rightarrow x - \ln K$ .

### 3. Auxiliary angle variable and matrix elements

#### 3.1. The $B_1$ algebra

The solutions to the  $K$  eigenvalue equation (2.7) found above form the basis for an IR of the group  $SO(2, 1)$ . These IR's are labelled by the value of  $K_0$ , and hence by the energy, while their rows determine the actual value of  $K$ . Computing matrix elements for different energies would then correspond to choosing two different IR's, rather than two rows of the same IR.

In the present section, we define a new algebra which has the virtue of exchanging these roles. It is based on an F-type factorisation of equation (2.6) (Infeld and Hull 1951). The explicit appearance of the quantum number  $n$  in this factorisation (Huffacker and Dwivedi 1975) suggests the introduction of a new variable  $\phi$  (Miller 1968) and the associated momentum:

$$G_0 = -i\partial/\partial\phi. \quad (3.1)$$

For vibrational states,  $\phi$  ranges from 0 to  $2\pi$ . Regarding the variable  $y$ , it is convenient to introduce an extra  $K$  factor:

$$r = K^2 e^{-x} \quad (\infty > r \geq 0). \quad (3.2)$$

We also define states in the  $(r, \phi)$  space, which are eigenstates of  $G_0$ , as

$$|\mu\rangle = \frac{1}{\sqrt{2\pi}} e^{i\mu\phi} \chi_\mu(r). \quad (3.3a)$$

$$\chi_\mu(r) = K e^{-x/2} S_\mu(x) \quad (3.3b)$$

is a solution of

$$\frac{d^2\chi_\mu(r)}{dr^2} + \left(\frac{2}{r} - \frac{\mu^2 - \frac{1}{4}}{r^2}\right)\chi_\mu(r) = K^{-2}\chi_\mu(r), \quad (3.4)$$

which follows directly from (2.6), and we have labelled  $\mu = K - n - \frac{1}{2}$ , according to (2.15). The functions  $\chi_\mu$  are orthonormal in the physical metric  $r^{-2} dr$ :

$$\int_0^\infty \chi_\mu^*(r)\chi_\mu(r)\frac{dr}{r^2} = \int_{-\infty}^\infty S_\mu^*(x)S_\nu(x) dx = \delta_{\mu\nu}. \tag{3.5}$$

The generalisation of the operator in (3.4) to the  $(r, \phi)$  space is simply

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{2}{r} - \frac{G_0^2 - \frac{1}{4}}{r^2}, \tag{3.6}$$

so that

$$\mathcal{L}|\mu\rangle = K^{-2}|\mu\rangle. \tag{3.7}$$

An important point to notice is that the operator  $\mathcal{L}$  is *not* self-adjoint in the physical metric defined in (3.5).

The factorisation of the operator  $\mathcal{L}$  is achieved using the two operators  $G_\pm$  (Herrick and Sinanoğlu 1972):

$$G_\pm = K e^{\pm i\phi} ((G_0 \pm \frac{1}{2})^2 r^{-1} \mp (G_0 \pm \frac{1}{2})(\partial/\partial r) - 1). \tag{3.8}$$

Indeed, using the relations

$$e^{\pm i\phi} (G_0 \pm \frac{1}{2}) = (G_0 \mp \frac{1}{2}) e^{\pm i\phi}, \tag{3.9}$$

we obtain

$$G_\pm G_\mp = K^2 [1 - (G_0 \mp \frac{1}{2})^2 \mathcal{L}]. \tag{3.10}$$

From the equations above we obtain the commutation relations

$$[G_0, G_\pm] = \pm G_\pm \quad [G_+, G_-] = 2G_0 I \tag{3.11}$$

which identify  $\{G_0, G_+, G_-\}$  as a realisation of a  $B_1$  algebra (Miller 1968). Here  $I$  is the identity operator in the space spanned by the kets  $|\mu\rangle$ , and  $G_\pm$  are the ladder operators:

$$G_\pm |\mu\rangle = G_\pm \frac{e^{i\mu\phi}}{\sqrt{2\pi}} \chi_\mu(r) = A_\mu^\pm \frac{e^{i(\mu\pm 1)\phi}}{\sqrt{2\pi}} \chi_{\mu\pm 1}(r) = A_\mu^\pm |\mu \pm 1\rangle. \tag{3.12}$$

The constants  $A_\mu^\pm$  are chosen so that the kets  $|\mu \pm 1\rangle$  remain normalised in the physical metric. They have to be computed taking into account the fact that  $G_+, G_-$  are *not* the adjoint of one another. Indeed, for the metric in (3.5), there is an additional term  $2/r$  when calculating the adjoint of  $\partial/\partial r$ , which results in

$$(G_\pm)^\dagger = G_\mp \mp 2Kr^{-1} e^{\mp i\phi} (G_0 \mp \frac{1}{2}) \tag{3.13}$$

when we take the adjoint of equation (3.8). The term in  $1/r$  above can be re-expressed in terms of the ladder operators as

$$r^{-1} = (G_0^2 - \frac{1}{4})^{-1} + \frac{1}{2} K^{-1} G_0^{-1} [(G_0 + \frac{1}{2})^{-1} e^{-i\phi} G_+ + (G_0 - \frac{1}{2})^{-1} e^{i\phi} G_-]. \tag{3.14}$$

As usual, to calculate the proportionality constants  $A_\mu^\pm$ , we evaluate the expectation value of  $G_\pm G_\mp$  in the  $(r, \phi)$  space, so using equations (3.7) and (3.10)–(3.14) we get

$$\langle \mu | G_\pm G_\mp | \mu \rangle = K^2 - (\mu \mp \frac{1}{2})^2 = |A_\mu^\mp|^2 (1 \mp \mu^{-1}). \tag{3.15}$$

The extra  $\mp \mu^{-1}$  term can be traced back directly to the extra term in (3.13). It does not appear in the usual angular momentum  $SO(3)$  algebra, where  $J_+$  is the adjoint of  $J_-$

(Edmonds 1957, Miller 1968). With the usual choice of phase, we finally obtain

$$G_{\pm}|\mu\rangle = A_{\mu}^{\pm}|\mu \pm 1\rangle = \left(\frac{\mu}{\mu \pm 1}\right)^{1/2} [K^2 - (\mu \pm \frac{1}{2})^2]^{1/2}|\mu \pm 1\rangle. \tag{3.16}$$

*3.2. The matrix elements*

In order to preserve the orthogonality of  $S_{\mu}(x)$ , we have introduced the metric  $d\phi r^{-2} dr$  for the kets  $|\mu\rangle$ . Hence, a knowledge of the coefficients  $A_{\mu}^{\pm}$  allows us to calculate matrix elements for operators expressed in terms of the generators of the algebra. In particular, since  $r^{-1}$  is a linear combination of  $G_+$  and  $G_-$ , we have a selection rule  $\nu = \mu, \mu \pm 1$  for this operator (Huffacker and Dwivedi 1975). Actually, using (3.14), we get

$$\langle \chi_{\nu} | r^{-1} | \chi_{\mu} \rangle = \langle \nu | e^{i(\nu-\mu)\phi} r^{-1} | \mu \rangle = \frac{4}{4\mu^2 - 1} \delta_{\nu\mu} + \frac{A_{\mu}^+}{K\mu(2\mu + 1)} \delta_{\nu\mu+1} + \frac{A_{\mu}^-}{K\mu(2\mu - 1)} \delta_{\nu\mu-1}, \tag{3.17}$$

with  $A_{\mu}^{\pm}$  given by (3.16), and  $\mu = K - n - \frac{1}{2}$  in terms of the number of quanta  $n$ . By the same token,

$$\langle \chi_{\nu} | d/dr | \chi_{\mu} \rangle = \frac{4}{4\mu^2 - 1} \delta_{\nu\mu} - \frac{(2\mu - 1)A_{\mu}^+}{2K\mu(2\mu + 1)} \delta_{\nu\mu+1} + \frac{(2\mu + 1)A_{\mu}^-}{2K\mu(2\mu - 1)} \delta_{\nu\mu-1}. \tag{3.18}$$

Next, we obtain a recurrence relation for the matrix elements of  $r^{\alpha}$  in terms of  $r^{\alpha+1}$ . To this end, we compute the commutator of  $\mathcal{L}$  with  $r^{\alpha+2}$ :

$$[\mathcal{L}, r^{\alpha+2}] = (\alpha + 1)(\alpha + 2)r^{\alpha} + 2(\alpha + 2)r^{\alpha+1}\partial/\partial r. \tag{3.19}$$

Using now the adjoint of  $\mathcal{L}$ ,

$$\mathcal{L}^{\dagger} = \mathcal{L} + 6r^{-2} - 4r^{-1}\partial/\partial r, \tag{3.20}$$

we calculate the matrix element of the commutator (3.19):

$$\langle \chi_{\nu} | [\mathcal{L}, r^{\alpha+2}] | \chi_{\mu} \rangle = \langle \nu | e^{i(\nu-\mu)\phi} [\mathcal{L}, r^{\alpha+2}] | \mu \rangle = \langle \nu | e^{i(\nu-\mu)\phi} (\mathcal{L}r^{\alpha+2} - r^{\alpha+2}\mathcal{L}) | \mu \rangle. \tag{3.21}$$

Substituting for  $\partial/\partial r$  the expression in terms of  $G_{\pm}$ , from their definition (3.8), we finally get the desired recurrence relation:

$$\left(\alpha + \frac{\mu^2 - \nu^2}{\alpha} + 4\mu^2\right) \langle \chi_{\nu} | r^{\alpha} | \chi_{\mu} \rangle = 4 \langle \chi_{\nu} | r^{\alpha+1} | \chi_{\mu} \rangle + \frac{2A_{\mu}^+}{K} \langle \chi_{\nu} | r^{\alpha+1} | \chi_{\mu+1} \rangle + \frac{2A_{\mu}^-}{K} \langle \chi_{\nu} | r^{\alpha+1} | \chi_{\mu-1} \rangle. \tag{3.22}$$

In particular, equation (3.17) follows from the above for  $\alpha = -1$ .

**4. The harmonic limit**

We now turn our attention to the case in which the Morse potential admits many bound states, and approaches a harmonic oscillator potential. Expanding (2.1) in powers of  $au$ , we get

$$W(u) \equiv D + V(u) = Da^2(u^2 - au^3 + \dots). \tag{4.1}$$

Letting  $a \rightarrow 0$  and  $D \rightarrow \infty$ , but keeping the force constant  $k_e = 2Da^2$  finite, the (inverse) anharmonicity parameter  $K \rightarrow \infty$ , the number of bound states increases, and we get the harmonic limit  $W = \frac{1}{2}k_e u^2$ .

We now want to study the contraction of the algebra  $\{G_0, G_+, G_-\}$  in this limit. For any finite value of the number of quanta  $n$ , we have that  $\mu \rightarrow K$ , and hence

$$G_0|\mu\rangle = \mu|\mu\rangle \rightarrow K|\mu\rangle, \tag{4.2}$$

so that  $G_0$  becomes the identity operator times  $K$ . On the other hand, expanding the exponential in (3.2), we get for  $G_{\pm}$

$$G_{\pm} \rightarrow \sqrt{2K} e^{\pm i\phi} \frac{1}{\sqrt{2}} \left( \pm \frac{1}{a\sqrt{K}} \frac{\partial}{\partial u} + a\sqrt{Ku} \right) \tag{4.3}$$

where  $a\sqrt{K} = \sqrt{m_0\omega/\hbar}$  remains finite in the harmonic limit. Hence the algebra  $\{G_0, G_+, G_-\}$  contracts into  $\{K.I, \sqrt{2K}A, \sqrt{2K}A^{\dagger}\}$ , where

$$[A, A^{\dagger}] = I. \tag{4.4}$$

This commutation relation is the limit of (3.11a), and the operators  $A^{\dagger}$  and  $A$  are the usual harmonic creation and annihilation operators. In this limit  $A$  and  $A^{\dagger}$  become the adjoint of one another, as they should, while (3.16) reduces to

$$A|n\rangle = \sqrt{n}|n-1\rangle \quad A^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{4.5}$$

The factor  $[\mu/(\mu \pm 1)]^{1/2}$  in (3.16) tends to one in this limit, since  $\mu \rightarrow \infty$  for any finite  $n$ .

### 5. Discussion

The  $SO(2, 1)$  algebra found in § 2 seems more appropriate for obtaining the discrete spectrum of the Morse oscillator. The spectrum is obtained by solving for the energy in terms of the anharmonicity parameter  $K$ . This in turn is the eigenvalue of the operator  $T_3$  in the algebra. To be able to calculate matrix elements between different energy eigenfunctions, it is necessary to introduce an additional variable and define tensor operators (Armstrong 1970). These tensors are not irreducible, but are indecomposable in this case (Chacón *et al* 1976). The matrix elements can thus be expressed in terms of the Wigner coefficients for the  $SO(2, 1)$  group (Holman and Biedenharn 1966).

The alternative route, followed in this paper, is to find an appropriate algebra whose ladder operators shift the value of the energy itself, rather than the potential parameters. The resulting Lie algebra is a realisation of a  $B_1$  algebra. The generators of this algebra do not present the usual adjointness properties of an  $SO(3)$  algebra when using the scalar product with physical meaning for the Morse oscillator. This, however, is not a real shortcoming for the computation of matrix elements for operators which are expressible in terms of the generators. This is done in a straightforward way without the need to define tensor operators.

These matrix elements can be used in perturbation expansions in which the Morse vibrational states are used as a starting point. These are utilised in spectroscopic problems for diatomic molecules (Huffacker 1976). In the case of dynamic problems involving the Morse oscillator, such as the problem of vibrational energy transfer for simple molecules (Rapp and Kassal 1969), or collinear reactive collisions (Hofacker and Levine 1971), a better approximation is usually required. Algebraic methods like



the ones developed in this paper allow for the evaluation of the corresponding transition probabilities (Levine and Wulfman 1979) in a non-perturbative fashion. Further work along these lines will be reported elsewhere.

### Acknowledgments

We would like to thank Professor R D Levine for very useful discussions on the subject.

### References

- Alhassid Y and Levine R D 1978 *Phys. Rev. A* **18** 89–116  
Anderson K L, Kumei S and Wulfman C E 1973 *J. Math. Phys.* **14** 1527–31  
Armstrong L 1970 *J. Physique* **31C4** 17–24  
— 1971 *Phys. Rev. A* **3** 1546–50  
Barut A O and Kleinert H 1967 *Phys. Rev.* **156** 1541–5  
Bednár M 1973 *Ann. Phys., NY* **75** 305–31  
Cabrera N, Celli V, Goodman F O and Hanson R 1970 *Surface Sci.* **19** 67–92  
Chacón E, Levi D and Moshinsky M 1976 *J. Math. Phys.* **17** 1919–29  
Čížek J and Paldus J 1977 *Int. J. Quantum Chem.* **12** 875–96  
Crubellier A and Feneuille S 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 1051–60  
Edmonds A R 1957 *Angular Momentum in Quantum Mechanics* (Princeton, NJ: Princeton University Press)  
Englefield M J 1972 *Group Theory and the Coulomb Problem* (New York: Wiley-Interscience)  
Herrick D and Sinanoğlu O 1972 *Phys. Rev. A* **5** 2309–13  
Hofacker G L and Levine R D 1971 *Chem. Phys. Lett.* **9** 617–20  
Holman W J and Biedenharn L C 1966 *Ann Phys., NY* **39** 1–42  
Huffacker J N 1976 *J. Chem. Phys.* **64** 3175–81  
Huffacker J N and Dwivedi P H 1975 *J. Math. Phys.* **16** 862–7  
Infeld L and Hull T 1951 *Rev. Mod. Phys.* **23** 21–68  
Levine R D and Wulfman C E 1979 *Chem. Phys. Lett.* **60** 372–6  
Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)  
Miller W 1968 *Lie Theory and Special Functions* (New York: Academic)  
Morse P M 1929 *Phys. Rev.* **34** 57–64  
Pauling L and Wilson E B 1935 *Introduction to Quantum Mechanics* (New York: McGraw-Hill)  
Rapp D and Kassal T 1969 *Chem. Rev.* **69** 61–102  
Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)